Jacobian has all eigenvalues with negative real part, even though the symmetric part is not negative-definite, is not even considered. (See Markus and Yamabe[2] for some results in this direction.) One problem in which the criterion verifies known results is that of how much damping is needed to insure stability in a problem such as the following. Consider

$$x'' + cx' + a(t)x = 0$$

where $a(t)$ is, say, periodic, positive, and bounded away from zero. The system can be unstable for $c = 0$ and for c small and positive. However, if c is large enough, the system is stable. The criterion of this paper will yield information on a problem of this nature.

—WARREN S. LOUD
Department of Mathematics
University of Minnesota

Conservation of the Form of the Maxwellian Distribution in a Relaxing Gas

D. I. OSIPOV

This paper investigates the establishment of Maxwellian equilib.ium in a light gas surrounded by an atmosphere of heavy gas. It is assumed that the interaction forces are inversely proportional to the fifth power of the distance. It is shown that if the initial distribution of the light gas is Maxwellian, at a temperature which differs from that of the heavy gas, the form of Maxwellian distribution in the light gas is conserved in the passage to equilibrium, and only the temperature changes. The extension of this result to other interaction laws is briefly considered.

IN THE relaxation of a gas to equilibrium the distribution function changes its form. If the initial distribution function is in equilibrium it may not be disturbed by relaxation under certain conditions. Thus in a system of harmonic oscillators dispersed in a thermostat, if the initial distribution of the oscillators over the vibrational levels is a Boltzman distribution (at a temperature differing from the temperature of the thermostat), the vibrational relaxation occurs in such a way that the distribution function of the oscillators remains strictly Boltzmannian, and the vibrational temperature alone changes (1).[1] The present paper examines another similar example—the conservation of the form of Maxwellian distribution in the process of relaxation.

Let us examine a system consisting of a large number of heavy atoms (of mass M) and a small number of light particles or electrons (of mass m). For simplicity's sake we shall speak henceforth of electrons alone. At the initial moment of time the temperature of the heavy particles (of the thermostat) is equal to T, and the temperature of the electrons is equal to T_0. We assume that the temperature of the thermostat remains constant during relaxation. We select the temperatures T and T_0 in a way that enables us to disregard all inelastic processes connected with the formation of negative and positive ions. Then the establishment of equilibrium in the electron gas will occur only as a result of energy exchange through elastic collisions of the electrons with the atoms of the thermostat.

The behavior of the electron distribution function $f(v,t)$ in the process of relaxation is described by the Boltzmann integral-differential equation. If we consider only elastic electron-atom collisions and assume that the electrons are distributed isotropically in velocity space, then, as shown in Ref. 2, the integral-differential equation becomes a differential equation of the Fokker-Planck type, having the form

$$\frac{\partial f}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}\left\{\frac{kT}{M}\frac{v^3}{\lambda(v)}\frac{\partial f}{\partial v} + \frac{mv^4}{M\lambda(v)}f\right\} \qquad [1]$$

[1] Numbers in parentheses indicate References at end of paper.

where λ is the length of the electron free path. Thus, as has also been done in Ref. 3, we assume that the effective electron-atom collision cross section is inversely proportional to the electron velocity, that is:

$$\lambda = \lambda_0\frac{v}{V} \qquad \text{where} \qquad V^2 = \frac{kT}{m}$$

This assumption is justified only for the Maxwellian law which assumes an interaction inversely proportional to the fifth power of the distance.

At the initial moment of time, the distribution function looks like

$$f(v,0) = N\left(\frac{m}{2\pi kT_0}\right)^{3/2}\exp\left[-\frac{mv^2}{2kT_0}\right] \qquad [2]$$

We solve Eq. [1] as follows:

$$f(v,t) = N\left(\frac{m}{2\pi k\vartheta(t)}\right)^{3/2}\exp\left[-\frac{mv^2}{2k\vartheta(t)}\right] \qquad [3]$$

where $\vartheta(t)$ is the unknown function determined by the initial condition

$$\vartheta(0) = T_0 \qquad [4]$$

Substituting [3] into [1] and abbreviating somewhat, we obtain for $\vartheta(t)$ the usual relaxation equation

$$\frac{d\vartheta}{dt} = -\frac{1}{\alpha}(\vartheta - T) \qquad [5]$$

the solution of which is

$$\vartheta - T = e^{-(t/\alpha)}(T_0 - T) \qquad [6]$$

where $\alpha = \lambda_0 M/2mV$ is the relaxation time. For an electron gas with $\lambda_0 = 6.5\cdot10^{-3}$ cm, located in a thermostat at a temperature $3kT/2 = 0.03$ ev and $M = 30(1832)m$, $\alpha = 4.06\cdot10^{-5}$ sec (3).

Of course, the field of application of the Maxwellian interaction potential is limited. Here the question arises of the conditions which, when used together with the interaction potential, will allow the form of the Maxwellian distribution

function to be conserved, at least in the first approximation. To obtain the appropriate criterion we make use of perturbation theory. If in solving Eq. [1] with arbitrary function $\lambda(v)$ for the unperturbed solution we take the Maxwellian function as it appears in [3] and consider as perturbation the right-hand side of [1], in which instead of $v/\lambda(v)$ we have $\{[v/\lambda(v)] - (V/\lambda_0)\}$, the condition for the application of the theory of perturbation may be written approximately as

$$\left| \frac{T_0 - T}{T_0} \frac{z - z_m}{z_m} \right| \ll 1 \qquad [7]$$

where $z = (v/\lambda)_{av}$, and $z_m = (V/\lambda_0) \cdot (v/\lambda)_{av}$ represents a temperature in the region (T_0, T), at which the second fraction in formula [7] is a maximum. This condition gives precisely the criterion needed to insure the conservation of the form of the Maxwellian distribution function with an arbitrary interaction potential.

It is evident from an examination of condition [7] that with small deviations from the equilibrium state the form of the Maxwellian function will be conserved with any reasonable intermolecular interaction potential, since $\max\left|(z - z_m)/z_m\right| \sim 1$. With an increase in the temperature deviation, the demands made on the potential become more severe.

The results obtained from this work are relevant for problems of kinetics, for example, in investigating the relaxation of the energy of translational degrees of freedom in a mixture of light and heavy gases heated by a shock wave.

In conclusion, the writer wishes to express his deep appreciation to Professor E. V. Stupochenko for discussing with him the results obtained.

References

1   Montroll, E. W. and Shuler, K. E., J. Chem. Phys. **25**, 454 (1957).
2   Chapman, S. and Cowling, T. G., *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1952).
3   Kahalas, S. L. and Kashian, H. C., Phys. Fluids **2**, 100 (1959).

Reviewer's Comment

This paper is of some interest in connection with two questions: 1) the general question of the relaxation of nonequilibrium subsystems in contact with a heat bath; and 2) the more specific question of the derivation of Fokker-Planck equations which describe the relaxation of nonequilibrium systems.

Osipov has pointed out that the maintenance of the Maxwell distribution in the relaxation of his ensemble of electrons dispersed in a heat bath of heavy particles is very reminiscent of the maintenance of the vibrational Boltzmann distribution in the relaxation of an ensemble of harmonic oscillators dispersed in an inert heat bath, as discussed previously by Montroll and Shuler. The general conditions under which initial Maxwell or Boltzmann distributions of a subsystem in an initial nonequilibrium distribution are maintained as Maxwell or Boltzmann distributions during the relaxation to equilibrium have never yet been worked out. A start toward this objective has been made by Shuler, Weiss, and Andersen in a paper recently submitted to the Journal of Mathematical Physics. In this paper, a derivation is given of the necessary and sufficient conditions for the exponential relaxation of the moments of distribution functions whose relaxation is described by a master equation or a Fokker-Planck equation. From these results, it was easy to show, without an explicit solution of Osipov's Fokker-Planck equation, that the translational temperature, which is proportional to the second moment, underwent an exponential relaxation (Osipov's Eqs. [5] and [6]) with a relaxation time $\alpha = \lambda_0 M/2mV$ as given by Osipov.

As to the second question raised above, the form of the Fokker-Planck equation to be used for the description of the time-dependent behavior of a Rayleigh gas presents some interesting problems that have not been touched on by Osipov.

Osipov, and Kahalas and Kashian (Osipov's Ref. 3) take over their Fokker-Planck equations directly from Chapman and Cowling. As has been emphasized again recently (see, e.g., Refs. 5 and 6), great care must be taken in the derivation of a Fokker-Planck equation so that the resulting equation is consistent in the order of the small expansion parameter, which, in the case of a Rayleigh gas, equals to m/M. For a Rayleigh gas in which a subsystem of heavy particles is dispersed in a heat bath of light particles, corresponding to the case of Brownian motion, a Fokker-Planck equation valid to a consistent order of the expansion parameter, $m/M \ll 1$ can be established readily as shown, for instance, by van Kampen in Ref. 5. For the case discussed by Osipov, however—i.e., a subsystem of light particles (electrons, for instance) dispersed in a heat bath of heavy particles—a rigorous derivation of the applicable Fokker-Planck has not yet been given, to the best knowledge of this reviewer. The case discussed by Osipov does not fall in the class of Brownian motion problems, since the light particle will suffer a *large* change of velocity in each collision with a heavy heat bath particle. It is not clear whether under these conditions a Fokker-Planck equation in *velocity space* will give a valid description of the time-dependent behavior of the relaxing subsystem of light particles. This is a question that requires some additional research.

—KURT E. SHULER
Institute for Defense Analyses
Washington, D. C.

1   Mazo, R. M., J. Chem. Phys. **35**, 831 (1961).
2   Uhlenbeck, G. E. and Wang Chang, C. S., *Proceedings of the International Symposium on Transport Processes in Statistical Mechanics, Brussels, 1956* (Interscience Publishers Inc., New York, 1958), p. 164.
3   Keilson, J. and Storer, J. E., Quart. J. Appl. Math. **10**, 243 (1952).
4   Shuler, K. E., Phys. Fluids **2**, 442 (1959).
5   van Kampen, N. G., Can. J. Phys. **30**, 551 (1961)
6   Siegel. A., J. Math. Phys. **1**. 378 (1960).